

Hidden Life of Riemann's Zeta Function

1. Arrow, Bow, and Targets

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Abstract

The Riemann Hypothesis is reformulated as statements about the eigenvalues of certain matrices entries of which are defined via the Taylor series coefficients of the zeta function. These eigenvalues demonstrate interesting visual patterns allowing one to state a number of conjectures.

1.1 The Hypothesis

One of the most interesting and important objects in number theory is Riemann's zeta function $\zeta(z)$. It can be defined for $\Re(z) > 1$ by the Dirichlet series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \quad (1.1.1)$$

The function can be extended to the entire complex z -plane with the exception of the point $z = 1$ which is the only pole of $\zeta(z)$. Points $z_1 = -2$, $z_2 = -4$, \dots , $z_n = -2n, \dots$ are known as the *trivial zeros* of the function $\zeta(z)$. We have the famous

Riemann Hypothesis (version 1). *All non-trivial zeros of the function $\zeta(z)$ lie on the critical line $\Re(z) = \frac{1}{2}$.*

1.2 Trivial zeroes

There is a tradition (taking its origin in Riemann's seminal paper [8]) to get rid of the trivial zeros by dealing with the function

$$\xi(z) = \pi^{-\frac{z}{2}}(z-1)\zeta(z)\Gamma(1+\frac{z}{2}) \quad (1.2.1)$$

rather than with the function $\zeta(z)$ itself (we use modern notation for this function, Riemann used $\xi(t)$ to denote the function which is today denoted $\Xi(t)$). The poles of the factor $\Gamma(1 + \frac{z}{2})$ in (1.2.1) cancel the trivial zeros of $\zeta(z)$ and similarly the factor $z - 1$ cancels the pole of $\zeta(z)$. The factor $\pi^{-\frac{z}{2}}$ influence neither zeros nor poles but it allows one to state the *functional equation* in a pretty form:

$$\xi(z) = \xi(1 - z). \quad (1.2.2)$$

In this paper we won't deprive zeta function of its trivial zeros but try to take advantage of our knowledge of precise positions of these zeros. To this end we will work with the entire function

$$\zeta^*(z) = 2(z - 1)\zeta(z). \quad (1.2.3)$$

For our purpose we could also omit the factor $z - 1$ and/or use the factor $\pi^{-\frac{z}{2}}$; this would change the picture(s) so probably separate paper(s) will be devoted to these variations. The factor 2 in (1.2.3) results in the equality

$$\zeta^*(0) = 1 \quad (1.2.4)$$

which slightly simplifies some forthcoming formulas.

According to (1.2.2) the non-trivial zeros of $\zeta(z)$ lie symmetrically around the critical line, so we have

Riemann Hypothesis (version 2). *The trivial zeros $z_1 = -2, z_2 = -4, \dots, z_n = -2n, \dots$ are the only zeros of the function $\zeta^*(z)$ lying in the half-plane $\Re(z) < \frac{1}{2}$.*

1.3 Change of the variable

A half-plane is a natural object when one deals with Dirichlet series. However, we are going to deal with Taylor series, and for them circles are more natural regions. So we make a change of variable:

$$z = \frac{w}{w + 1}, \quad w = \frac{z}{1 - z}. \quad (1.3.1)$$

Under this transformation the critical line becomes the *critical circle* $|w| = 1$, the half-plane $\Re(z) < \frac{1}{2}$ becomes the interior of this circle, and points

$$w_1 = \frac{z_1}{1 - z_1} = -\frac{2}{3}, \dots, w_n = \frac{z_n}{1 - z_n} = -\frac{2n}{2n + 1}, \dots \quad (1.3.2)$$

become the *trivial zeros of the function*

$$\tilde{\zeta}(w) = \zeta^*\left(\frac{w}{w + 1}\right). \quad (1.3.3)$$

With this new notation we have

Riemann Hypothesis (version 3). *The trivial zeros $w_1 = -\frac{2}{3}, \dots, w_n = -\frac{2n}{2n + 1}, \dots$ are the only zeros of the function $\tilde{\zeta}(w)$ lying in the open circle $|w| < 1$.*

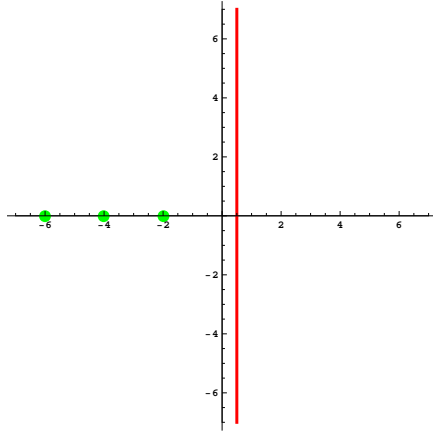


Figure 1.3.1: z -plane

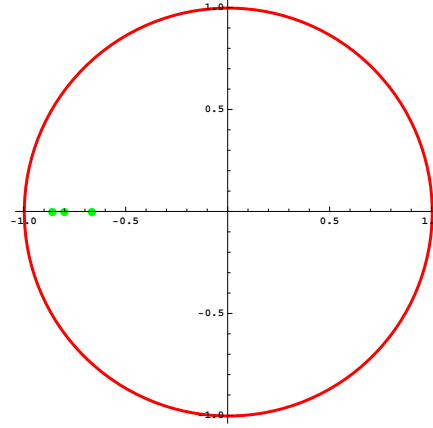


Figure 1.3.2: w -plane

1.4 Subhypotheses

It isn't very convenient to work near the critical circle (full of zeros) so we split the Riemann Hypothesis into an infinite series of weaker statements:

RH_l, the l -th Riemann subhypothesis (version 1). *The trivial zeros $w_1 = -\frac{2}{3}, \dots, w_l = -\frac{2l}{2l+1}$ are the only zeros of the function $\tilde{\zeta}(w)$ lying in the closed disk $|w| \leq \frac{2l+1}{2l+2}$.*

While each of the subhypotheses is weaker than the Riemann Hypothesis, taken together, they are equivalent to it:

Riemann Hypothesis (version 4). *For every m the subhypothesis RH_m is true.*

1.5 First question

Let us ask a "naïve" question: *Why is RH₁ true?* More precisely, *how can we see that RH₁ is true?* To answer this question we can expand the function $1/\tilde{\zeta}(w)$ into the Taylor series:

$$1/\tilde{\zeta}(w) = 1 + \tau_1 w + \dots + \tau_n w^n + \dots \quad (1.5.1)$$

Now according to RH₁ the point $w_1 = -\frac{2}{3}$ should be the only pole of the function (1.5.1) lying inside the circle $|w| \leq \frac{3}{4}$. Respectively, we have

RH₁ (version 2). *For $m \rightarrow \infty$*

$$\tau_m = \left(-\frac{3}{2}\right)^m (R_1 + o(1)) \quad (1.5.2)$$

for some non-zero constant R_1 .

It is easy to see that

$$R_1 = \frac{3}{2\tilde{\zeta}'(-2/3)} \quad (1.5.3)$$

$$= -\frac{1}{36\zeta'(-2)} \quad (1.5.4)$$

$$= 0.91228851841347\dots \quad (1.5.5)$$

$$> 0 \quad (1.5.6)$$

so we have

RH₁ (version 3).

$$\lim_{m \rightarrow \infty} ((-1)^m \tau_m)^{\frac{1}{m}} = \frac{3}{2}. \quad (1.5.7)$$

1.6 Determinant representation

It is easy to see that coefficients τ_1, τ_2, \dots from (1.5.1) can be expressed in terms of the coefficients in the Taylor expansion

$$\tilde{\zeta}(w) = 1 + \theta_1 w + \dots + \theta_m w^m + \dots \quad (1.6.1)$$

More precisely,

$$\tau_m = (-1)^m \det(L_{1,m}) \quad (1.6.2)$$

where $L_{1,m}$ is the following Toeplitz matrix:¹

$$L_{1,m} = \begin{pmatrix} \theta_1 & 1 & 0 & \dots & 0 & 0 \\ \theta_2 & \theta_1 & 1 & \dots & 0 & 0 \\ \theta_3 & \theta_2 & \theta_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta_{m-1} & \theta_{m-2} & \theta_{m-3} & \dots & \theta_1 & 1 \\ \theta_m & \theta_{m-1} & \theta_{m-2} & \dots & \theta_2 & \theta_1 \end{pmatrix}. \quad (1.6.3)$$

Then, we have

RH₁ (version 4). For $m \rightarrow \infty$

$$\det(L_{1,m}) = \left(-\frac{3}{2}\right)^m (R_1 + o(1)) \quad (1.6.4)$$

with the constant R_1 defined by (1.5.3)–(1.5.4)

¹Clearly, we can change the order of the columns in this matrix and obtain a Hankel matrix whose determinant has the same absolute value; the resulting picture(s) are rather different, they are considered in [7]; for our purpose we can also deal with the product of our matrix $L_{1,m}$ and its Hankel counterpart.

and

RH₁ (version 5).

$$\lim_{m \rightarrow \infty} (\det(L_{1,m}))^{\frac{1}{m}} = \frac{3}{2}. \quad (1.6.5)$$

1.7 Eigenvalues on average

Naturally,

$$\det(L_{1,m}) = \lambda_{1,m,1} \lambda_{1,m,2} \cdots \lambda_{1,m,m} \quad (1.7.1)$$

where $\lambda_{1,m,1}, \lambda_{1,m,2}, \dots, \lambda_{1,m,m}$ are the eigenvalues of the matrix $L_{1,m}$. Thus, we have

RH₁ (version 6).

$$\lim_{m \rightarrow \infty} \left(\prod_{n=1}^m \lambda_{1,m,n} \right)^{\frac{1}{m}} = \frac{3}{2}. \quad (1.7.2)$$

The (multi)set $\{\lambda_{1,m,1}, \lambda_{1,m,2}, \dots, \lambda_{1,m,m}\}$ will be called the λ -*spectrum* and will be denoted $\text{Spec}_{1,m}^\lambda$.

1.8 Positions of individual eigenvalues

According to RH₁ the (geometric) mean of $\lambda_{1,m,1}, \lambda_{1,m,2}, \dots, \lambda_{1,m,m}$ approaches $\frac{3}{2}$ when m goes to infinity, but neither RH₁ nor RH itself tells us anything directly about the distribution of these eigenvalues. Are they as random as, say, the imaginary parts of the non-trivial zeros of $\zeta(z)$? Do the eigenvalues cluster or are they spread around the whole w -plane? Is there any similarity between eigenvalues corresponding to different values of m ?

The author was curious to calculate² the values of the spectra $\text{Spec}_{1,m}^\lambda$ for initial values of m and have a look at them. Some pictures are included in this paper, an updated collection of pictures can be downloaded from [6].

Figures 1.8.1–1.8.4 show spectra $\text{Spec}_{1,m}^\lambda$ for $m = 24, 48, 96, 192$ respectively together with the circle $|w| = \frac{3}{2}$, the “ideal” place for the eigenvalues according to (1.7.2). Figure 1.8.5 shows the union $\cup_{m=1}^{192} \text{Spec}_{1,m}^\lambda$ (in the first quadrant). The “hidden life of Riemann’s zeta function” is best seen from an animation showing the $\text{Spec}_{1,1}^\lambda, \text{Spec}_{1,2}^\lambda, \dots$ in succession; such an animation can be downloaded from [6].

Looking at the pictures we can say that the λ -spectrum $\text{Spec}_{1,m}^\lambda$ is the union of the *arrow* $\text{Arr}_{1,m}$ consisting entirely of real eigenvalues and the *bow* $\text{Bow}_{1,m}$; for counting purpose it is reasonable to consider sometimes the largest real

²Calculations were done mainly with MATHEMATICA and partly with PARI on a personal computer; larger scale computations are very desirable for getting more insight.

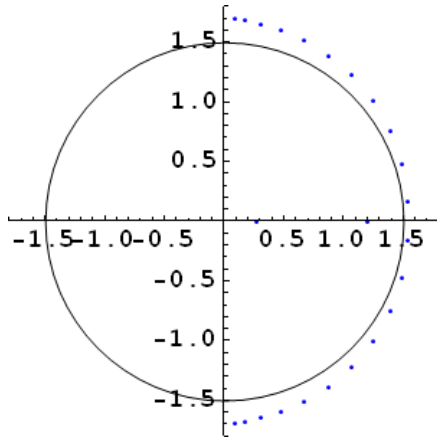


Figure 1.8.1: $\text{Spec}_{1,24}^\lambda$

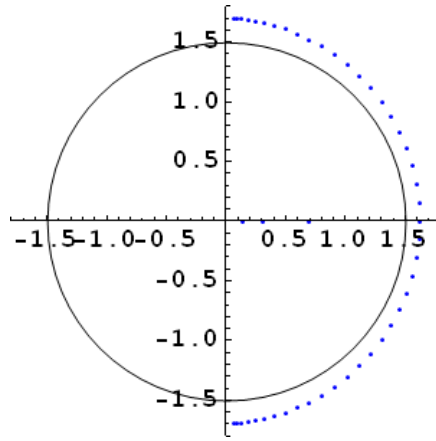


Figure 1.8.2: $\text{Spec}_{1,48}^\lambda$

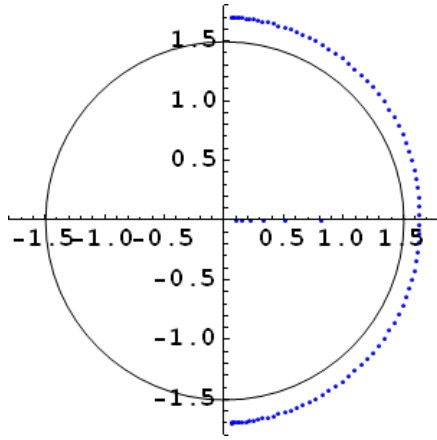


Figure 1.8.3: $\text{Spec}_{1,96}^\lambda$

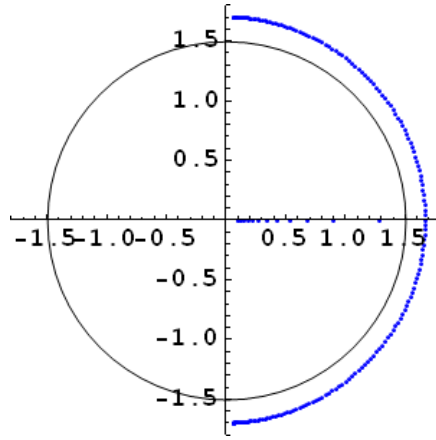


Figure 1.8.4: $\text{Spec}_{1,192}^\lambda$

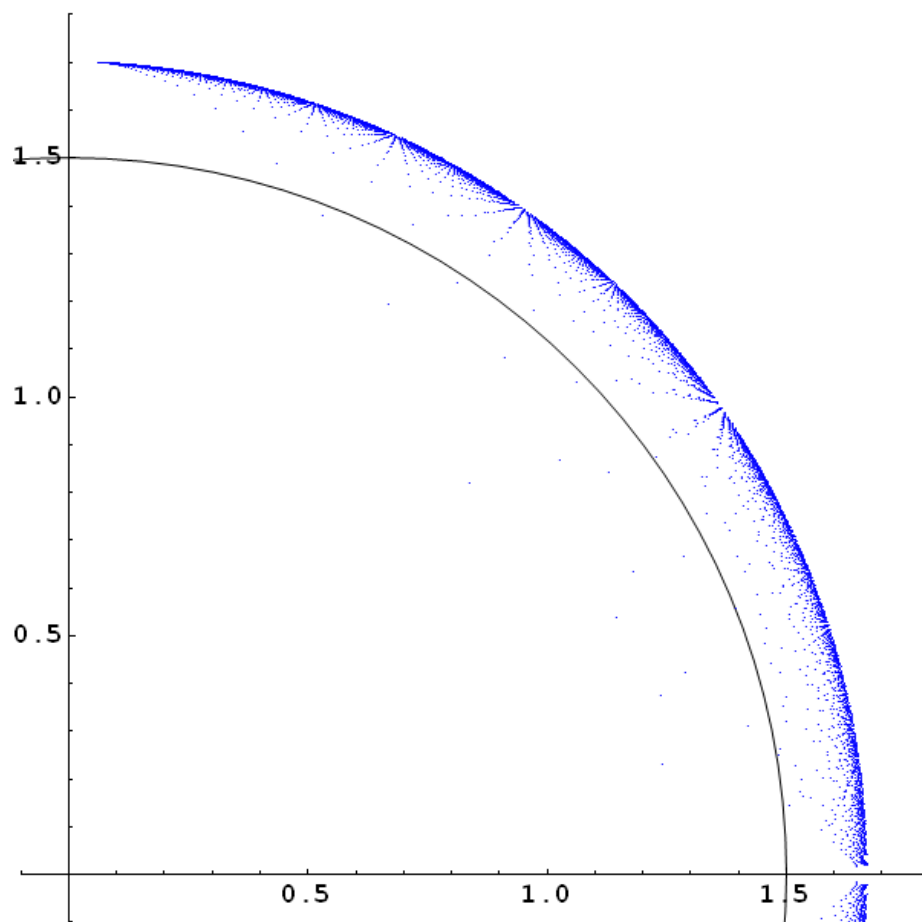


Figure 1.8.5: $\cup_{m=1}^{192} \text{Spec}_{1,m}^\lambda$

eigenvalue as belonging to the bow rather than to the arrow. Formally, if the number of real eigenvalues in the spectrum $\text{Spec}_{1,m+1}^\lambda$ is less than the number of real eigenvalues in the spectrum $\text{Spec}_{1,m}^\lambda$, then we consider the largest real eigenvalue in the spectrum $\text{Spec}_{1,m}^\lambda$ as belonging to $\text{Bow}_{1,m}$ and not belonging to arrow $\text{Arr}_{1,m}$.

1.9 First Conjectures

The above pictures suggest the following conjectures.

Conjecture 1A₁. *There are no multiple eigenvalues.*

Conjecture 1B₁. $\sup_m(\max(\text{Arr}_{1,m}))$ *is a positive number.*

Conjecture 1C₁. $\inf_m(\min(\text{Arr}_{1,m}))$ *is a positive number.*

Conjecture 1D₁. *The numbers $\text{arr}_{1,m} = \|\text{Arr}_{1,m}\|$ and $\text{bow}_{1,m} = \|\text{Bow}_{1,m}\|$ of eigenvalues belonging to the arrow $\text{Arr}_{1,m}$ and to the bow $\text{Bow}_{1,m}$ respectively don't decrease when m increases.*

Conjecture 1E₁. *If*

$$\text{Arr}_{1,m} = \{\lambda_{1,m,1}, \lambda_{1,m,2}, \dots, \lambda_{1,m,\text{arr}_{1,m}}\}, \quad (1.9.1)$$

$$\text{Arr}_{1,m+1} = \{\lambda_{1,m+1,1}, \lambda_{1,m+1,2}, \dots, \lambda_{1,m+1,\text{arr}_{1,m+1}}\} \quad (1.9.2)$$

and

$$\lambda_{1,m,1} < \lambda_{1,m,2} < \dots < \lambda_{1,m,\text{arr}_{1,m}}, \quad (1.9.3)$$

$$\lambda_{1,m+1,1} < \lambda_{1,m+1,2} < \dots < \lambda_{1,m+1,\text{arr}_{1,m+1}} \quad (1.9.4)$$

then

$$\lambda_{1,m+1,1} < \lambda_{1,m,1}, \quad \lambda_{1,m+1,2} < \lambda_{1,m,2}, \dots, \lambda_{1,m+1,\text{arr}_{1,m}} < \lambda_{1,m,\text{arr}_{1,m}}. \quad (1.9.5)$$

Conjecture 1F₁. *Assign the weight $\frac{1}{m}$ to each of the points $\lambda_{1,m,1}, \lambda_{1,m,2}, \dots, \lambda_{1,m,m}$ and denote by $\lambda_{1,m}$ corresponding discrete measure. Then*

1F'₁. *there exists a limiting continuous measure $\lambda_1(w)$ concentrated on a “limiting bow” and a “limiting arrow”;*

1F''₁. $\int \log(w) d\lambda_1(w) = \log\left(\frac{3}{2}\right).$

Clearly, Conjecture 1F₁ implies Subhypothesis RH₁.

1.10 Purely Trivial Zeros

It is natural to try to understand to what extent the distribution of $\lambda_{1,m,1}$, $\lambda_{1,m,2}$, \dots , $\lambda_{1,m,m}$ is due to the trivial zeros, and what is the contribution of the non-trivial zeros. To this end we can consider the function

$$\zeta_T(z) = \frac{\zeta^*(z)}{2\xi(z)} \quad (1.10.1)$$

$$= \frac{\pi^{\frac{z}{2}}}{\Gamma(1 + \frac{z}{2})}. \quad (1.10.2)$$

The points $z_1 = -2$, $z_2 = -4$, \dots , $z_n = -2n, \dots$ are the only zeros of the function $\zeta_T(z)$. The factor 2 in the denominator of (1.10.1) implies the equality

$$\zeta_T(0) = 1 \quad (1.10.3)$$

analogous to the equality (1.2.4).

By analogy with (1.3.3) and (1.6.1), for every analytic function $f(z)$ such that

$$f(0) = 1 \quad (1.10.4)$$

we can consider the transformed function

$$\tilde{f}(w) = f\left(\frac{w}{1+w}\right) \quad (1.10.5)$$

with the expansion

$$\tilde{f}(w) = 1 + \theta_1(f)w + \dots + \theta_m(f)w^m + \dots, \quad (1.10.6)$$

form the matrices $L_{1,m}(f)$, counterparts of (1.6.3), with eigenvalues $\lambda_{1,m,1}(f)$, $\lambda_{1,m,2}(f) \dots$, $\lambda_{1,m,m}(f)$, and state various versions of subhypothesis $\text{RH}_1(f)$.

Figures 1.10.1–1.10.4 show spectra $\text{Spec}_{1,m}^\lambda(\zeta_T)$ in black color together with $\text{Spec}_{1,m}^\lambda(\zeta^*)$ in blue color for $m = 24, 48, 96, 192$ respectively. These figures suggest that the distribution of the λ 's is to a great extent determined by the trivial zeros.

The gamma function is supposed to be “simple”, “completely understood”, a function about which we know everything; it would be natural, as a first step towards the Riemann Hypothesis, to understand the character of the numbers $\lambda_{1,m,n}(\zeta_T)$.

1.11 Further Questions

Now, how could we see that RH_2 , RH_3 , \dots are true? The Taylor expansion (1.5.1) doesn't tell us anything directly about the other poles of the function $1/\tilde{\zeta}(w)$. One way to overcome this obstacle could be to consider the function

$$\hat{\zeta}_l(z) = \frac{\zeta^*(z)}{\prod_{k=1}^{l-1} (1 - \frac{z}{z_k})}; \quad (1.11.1)$$

a separate paper may be devoted to the corresponding eigenvalues $\lambda_{1,m,1}(\hat{\zeta}_l)$, $\lambda_{1,m,2}(\hat{\zeta}_l) \dots$, $\lambda_{1,m,m}(\hat{\zeta}_l)$.

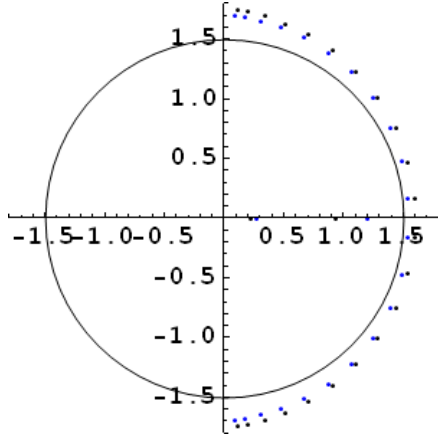


Figure 1.10.1: $\text{Spec}_{1,24}^\lambda(\zeta^*)$ and $\text{Spec}_{1,24}^\lambda(\zeta_T)$

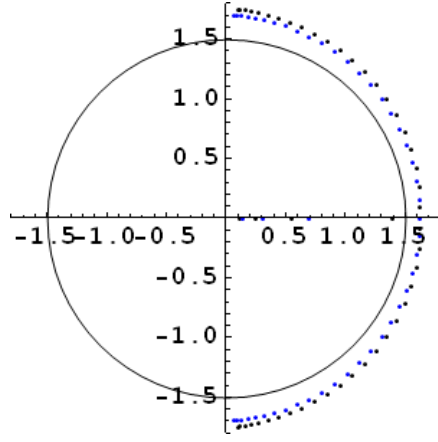


Figure 1.10.2: $\text{Spec}_{1,48}^\lambda(\zeta^*)$ and $\text{Spec}_{1,48}^\lambda(\zeta_T)$

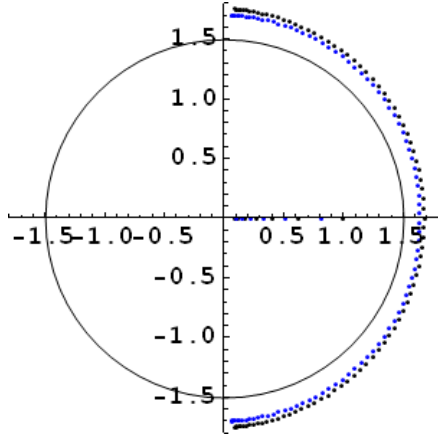


Figure 1.10.3: $\text{Spec}_{1,96}^\lambda(\zeta^*)$ and $\text{Spec}_{1,96}^\lambda(\zeta_T)$

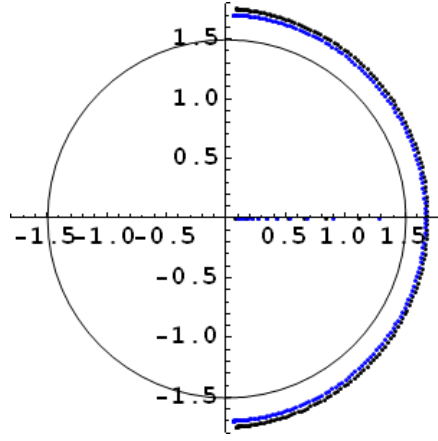


Figure 1.10.4: $\text{Spec}_{1,192}^\lambda(\zeta^*)$ and $\text{Spec}_{1,192}^\lambda(\zeta_T)$

1.12 Padé approximations

However, another approach to RH_m for arbitrary m looks more promising. This approach treats the first m trivial zeros “on equal” and is based on Padé approximations.

To begin with, let $P_{1,m}(w)$ and $Q_{1,m}(w)$ be polynomials such that

$$\tilde{\zeta}(w) \approx \frac{P_{1,m}(w)}{Q_{1,m}(w)} = \frac{1 + p_{1,m,1}w}{1 + q_{1,m,1}w + \cdots + q_{1,m,m}w^m} \quad (1.12.1)$$

$$= \tilde{\zeta}(w) + O(w^{m+2}) \quad (1.12.2)$$

It can be checked that

$$p_{1,m,1} = -\frac{\tau_{m+1}}{\tau_m} \quad (1.12.3)$$

and hence, according to (1.5.2), for $m \rightarrow \infty$

$$p_{1,m,1} \rightarrow \frac{3}{2} \quad (1.12.4)$$

and, respectively,

$$P_{1,m}(w) \rightarrow 1 + \frac{3}{2}w = 1 - \frac{w}{w_1}. \quad (1.12.5)$$

Now let us consider the general case. Let $P_{l,m}(w)$ and $Q_{l,m}(w)$ be such polynomials that

$$\tilde{\zeta}(w) \approx \frac{P_{l,m}(w)}{Q_{l,m}(w)} = \frac{1 + p_{l,m,1}w + \cdots + p_{l,m,l}w^l}{1 + q_{l,m,1}w + \cdots + q_{l,m,m}w^m} \quad (1.12.6)$$

$$= \tilde{\zeta}(w) + O(w^{l+m+1}) \quad (1.12.7)$$

According to a theorem of de Montessus [2, 3] (see also [1]) for every l , subhypothesis RH_l implies the following generalization of (1.12.5): *For all l for $m \rightarrow \infty$*

$$P_{l,m}(w) \rightarrow \prod_{k=1}^l \left(1 - \frac{w}{w_l}\right). \quad (1.12.8)$$

We are going to deal only with the leading coefficient of $P_{l,m}(w)$ for which (1.12.8) implies the following generalization of (1.12.4):

Subhypothesis RH_l^w (version 1). *For $m \rightarrow \infty$*

$$p_{l,m,l} \rightarrow W_l \quad (1.12.9)$$

where

$$W_l = \prod_{k=1}^l \left(-\frac{1}{w_l}\right) = \prod_{k=1}^l \frac{2k+1}{2k}. \quad (1.12.10)$$

1.13 Back to the Riemann Hypothesis

Each subhypothesis RH_l^w is, formally, weaker than the corresponding subhypothesis RH_l , nevertheless, taken together the subhypotheses RH_l^w are equivalent to the subhypotheses RH_l , and thus we have

Riemann Hypothesis (version 5). *For every m the subhypothesis RH_l^w is true.*

In order to see why it is so, suppose that the Riemann Hypothesis isn't valid, and let \check{z} be a non-trivial zero of $\zeta(z)$ with $\Re(\check{z}) < \frac{1}{2}$. Then $\check{w} = \frac{\check{z}}{1-\check{z}}$ is a non-trivial zero of $\tilde{\zeta}(w)$ with $|\check{w}| < 1$. In the closed circle $|w| \leq |\check{w}|$ there are only finitely many zeros of $\tilde{\zeta}(w)$; let us denote them by $\check{w}_1, \dots, \check{w}_l$. By the above cited theorem of de Montessus, for $m \rightarrow \infty$

$$P_{l,m}(w) \rightarrow \prod_{k=1}^l \left(1 - \frac{w}{\check{w}_l}\right) \quad (1.13.1)$$

and hence

$$p_{l,m,l} \rightarrow \prod_{k=1}^l \left(-\frac{1}{\check{w}_l}\right). \quad (1.13.2)$$

It is easy to see that

$$\left| \prod_{k=1}^l \left(-\frac{1}{\check{w}_l}\right) \right| > \left| \prod_{k=1}^l \left(-\frac{1}{w_l}\right) \right| = |W_l| \quad (1.13.3)$$

which gives the required contradiction with (1.12.9).

1.14 More Determinants

An explicit expression for $p_{l,m,l}$ can be given (Jacobi [4], see also [1]):

$$p_{l,m,l} = \frac{\det(L_{l,m+1}(\zeta^*))}{\det(L_{l,m}(\zeta^*))} \quad (1.14.1)$$

where

$$L_{l,m}(f) = \begin{pmatrix} \theta_l(f) & \theta_{l-1}(f) & \dots & \theta_{l-m+1}(f) \\ \theta_{l+1}(f) & \theta_l(f) & \dots & \theta_{l-m+2}(f) \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{l+m-1}(f) & \theta_{l+m-2}(f) & \dots & \theta_l(f) \end{pmatrix} \quad (1.14.2)$$

with $\theta_0(f) = 1$ and $\theta_j(f) = 0$ for $j < 0$.

In terms of these matrices we have the following counterpart of (1.6.4):

RH_l^w (version 2). For $m \rightarrow \infty$

$$\det(L_{l,m}(\zeta^*)) = W_l^m(R_l(\zeta^*) + o(1)). \quad (1.14.3)$$

with some constant $R_l(\zeta^*)$.

In order to pass from (1.5.2), the second version of RH₁, to (1.5.7), the third version of RH₁, we needed the inequality (1.5.6) which we got from the numerical value (1.5.5). However, it is easy to see that RH₁(f) implies the inequality $R_1(f) > 0$ for every function f satisfying condition (1.10.4). Namely, by analogy with (1.5.3) $\text{RH}_1(f) = -\frac{1}{z_1 f'(z_1)}$ and $f'(z_1) > 0$ because $z_1 = -\frac{2}{3}$ is the least (in absolute value) zero of $f(z)$. By a similar argument, for every l the inequality $R_l(f) > 0$ is implied by RH_l(f), and we have

RH_l^w (version 3).

$$\lim_{m \rightarrow \infty} (\det(L_{l,m}(\zeta^*)))^{\frac{1}{m}} = W_l. \quad (1.14.4)$$

1.15 More Eigenvalues

By analogy with (1.7.1), we have the representation

$$\det(L_{l,m}(f)) = \lambda_{l,m,1}(f) \lambda_{l,m,2}(f) \dots \lambda_{l,m,m}(f) \quad (1.15.1)$$

where $\lambda_{l,m,1}(f)$, $\lambda_{l,m,2}(f)$, \dots , $\lambda_{l,m,m}(f)$ are the eigenvalues of the matrix $L_{l,m}(f)$. Then, we have

RH_l^w (version 4).

$$\lim_{m \rightarrow \infty} \left(\prod_{n=1}^m \lambda_{l,m,n}(\zeta^*) \right)^{\frac{1}{m}} = W_l. \quad (1.15.2)$$

The (multi)set $\{\lambda_{l,m,1}(f), \lambda_{l,m,2}(f), \dots, \lambda_{l,m,m}(f)\}$ will be called the λ -*spectrum* of the function f and will be denoted $\text{Spec}_{l,m}^\lambda(f)$.

1.16 More about positions of eigenvalues

Figures 1.16.1–1.16.4 show spectra $\text{Spec}_{2,m}^\lambda(\zeta^*)$ for $m = 24, 48, 96, 192$ respectively. An animation showing the $\text{Spec}_{2,1}^\lambda(\zeta^*)$, $\text{Spec}_{2,2}^\lambda(\zeta^*)$, \dots in succession can be downloaded from [6].

We see that $\text{Spec}_{2,m}^\lambda(\zeta^*)$ consists of the arrow, the bow (now looking into the opposite direction), and a new element, looking like a circle, which will be called *orbit*. The animation shows that the orbit has, on its right-hand side, a

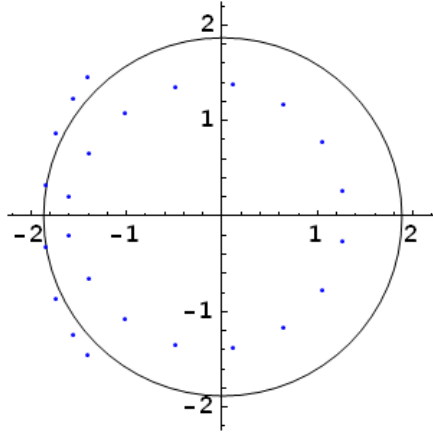


Figure 1.16.1: $\text{Spec}_{2,24}^\lambda(\zeta^*)$

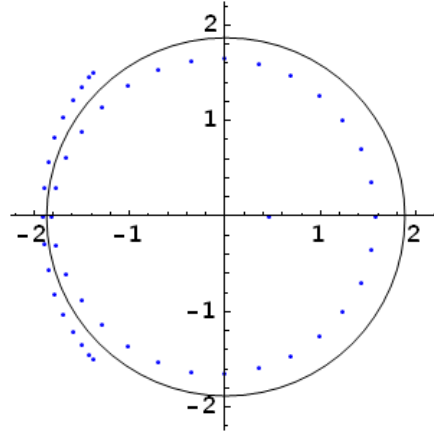


Figure 1.16.2: $\text{Spec}_{2,48}^\lambda(\zeta^*)$

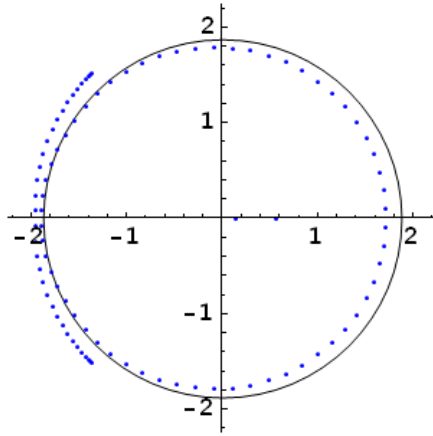


Figure 1.16.3: $\text{Spec}_{2,96}^\lambda(\zeta^*)$

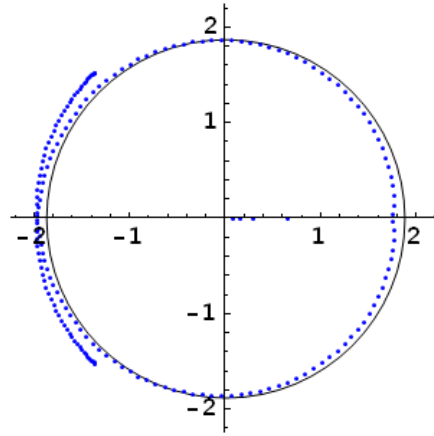


Figure 1.16.4: $\text{Spec}_{2,192}^\lambda(\zeta^*)$

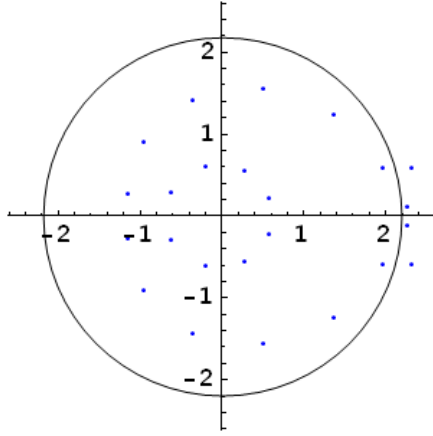


Figure 1.16.5: $\text{Spec}_{3,24}^\lambda(\zeta^*)$

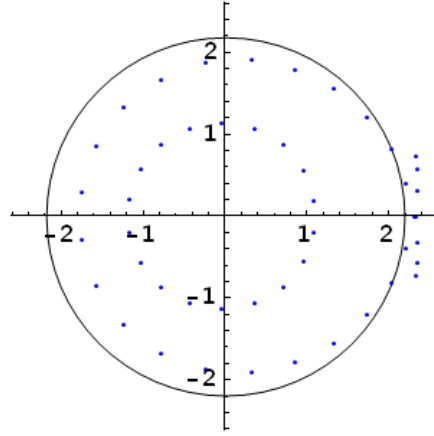


Figure 1.16.6: $\text{Spec}_{3,48}^\lambda(\zeta^*)$

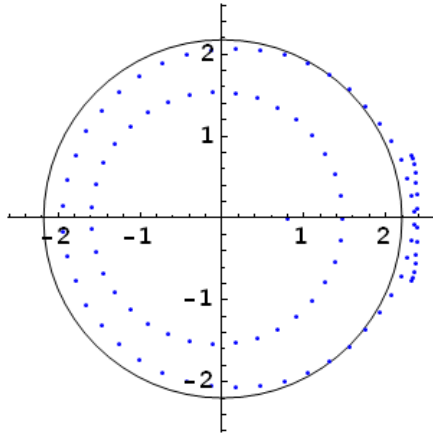


Figure 1.16.7: $\text{Spec}_{3,96}^\lambda(\zeta^*)$

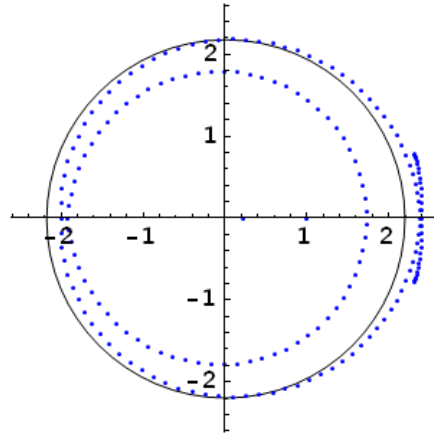


Figure 1.16.8: $\text{Spec}_{3,192}^\lambda(\zeta^*)$

rendezvous with the arrow and, on its left-hand side, another *rendezvous* with the bow.

Figures 1.16.5–1.16.8 show spectra $\text{Spec}_{3,m}^\lambda(\zeta^*)$ for $m = 24, 48, 96, 192$ respectively. An animation showing the $\text{Spec}_{3,1}^\lambda(\zeta^*)$, $\text{Spec}_{3,2}^\lambda(\zeta^*)$, ... in succession can be downloaded from [6].

We see that $\text{Spec}_{3,m}^\lambda(\zeta^*)$ consists of the arrow, the bow (now rather rudimentary and looking into the same direction as in the case of $\text{Spec}_{1,m}^\lambda$), and two orbits which constitute *target*. The animation shows that the inner orbit has, on its right-hand side, the rendezvous with the arrow and has, on its left-hand side, the rendezvous with the outer orbit. In its turn, the outer orbit has, on its left-hand side, the rendezvous with the inner orbit and has, on its right-hand side, the rendezvous with the bow.

One might expect that the target of the spectra $\text{Spec}_{4,m}^\lambda(\zeta^*)$ would consist of three orbits but this is not the case. Figures 1.16.9–1.16.12 show spectra $\text{Spec}_{4,m}^\lambda(\zeta^*)$ for $m = 24, 48, 96, 192$ respectively. An animation showing the $\text{Spec}_{4,1}^\lambda(\zeta^*)$, $\text{Spec}_{4,2}^\lambda(\zeta^*)$, ... in succession can be downloaded from [6]. We see that the entire structure of the spectra $\text{Spec}_{4,m}^\lambda(\zeta^*)$ is the same as in the case of the spectra $\text{Spec}_{3,m}^\lambda(\zeta^*)$ but the bow isn't rudimentary any longer, on the contrary, it is almost a circle.

The third orbit in the target appears in the spectra $\text{Spec}_{5,m}^\lambda(\zeta^*)$. Figures 1.16.13–1.16.16 show spectra $\text{Spec}_{5,m}^\lambda(\zeta^*)$ for $m = 24, 48, 96, 192$ respectively. An animation showing the $\text{Spec}_{5,1}^\lambda(\zeta^*)$, $\text{Spec}_{5,2}^\lambda(\zeta^*)$, ... in succession can be downloaded from [6].

Similar, the fourth orbit in the target appears in the spectra $\text{Spec}_{6,m}^\lambda(\zeta^*)$. Figures 1.16.17–1.16.20 show spectra $\text{Spec}_{6,m}^\lambda(\zeta^*)$ for $m = 24, 48, 96, 192$ respectively. An animation showing the $\text{Spec}_{6,1}^\lambda(\zeta^*)$, $\text{Spec}_{6,2}^\lambda(\zeta^*)$, ... in succession can be downloaded from [6].

The fifth orbit in the target appears in the spectra $\text{Spec}_{7,m}^\lambda(\zeta^*)$. Figures 1.16.21–1.16.24 show spectra $\text{Spec}_{7,m}^\lambda(\zeta^*)$ for $m = 24, 48, 96, 192$ respectively. An animation showing the $\text{Spec}_{7,1}^\lambda(\zeta^*)$, $\text{Spec}_{7,2}^\lambda(\zeta^*)$, ... in succession can be downloaded from [6].

The above pictures don't show the arrows in any of $\text{Spec}_{5,m}^\lambda(\zeta^*)$, $\text{Spec}_{6,m}^\lambda(\zeta^*)$, $\text{Spec}_{7,m}^\lambda(\zeta^*)$ for $m = 24, 48, 96, 192$ but probably the arrows will appear for sufficiently large m .

1.17 More Conjectures

The above pictures suggest the following conjectures which, in particular, generalize conjectures 1A₁–1F₁.

Conjecture 1A. *There are never multiple eigenvalues in $\text{Spec}_{l,m}^\lambda(\zeta^*)$.*

Conjecture 1B. *For all l , $\sup_m(\max(\text{Arr}_{l,m}(\zeta^*)))$ is a positive number.*

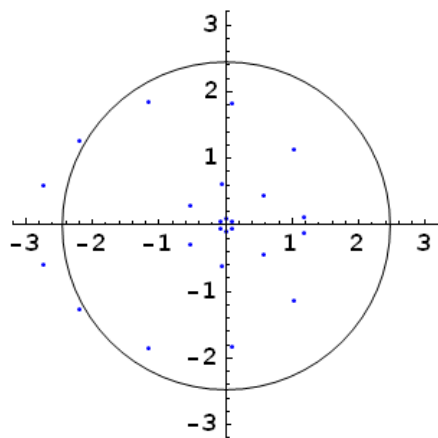


Figure 1.16.9: $\text{Spec}_{4,24}^{\lambda}(\zeta^*)$

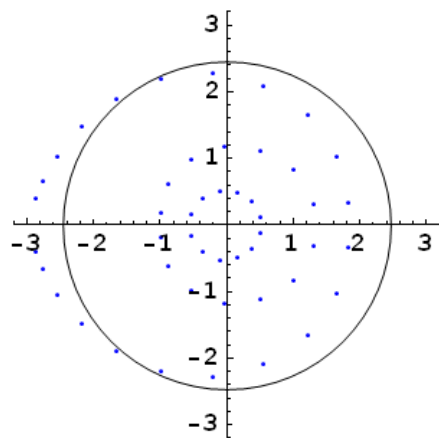


Figure 1.16.10: $\text{Spec}_{4,48}^{\lambda}(\zeta^*)$

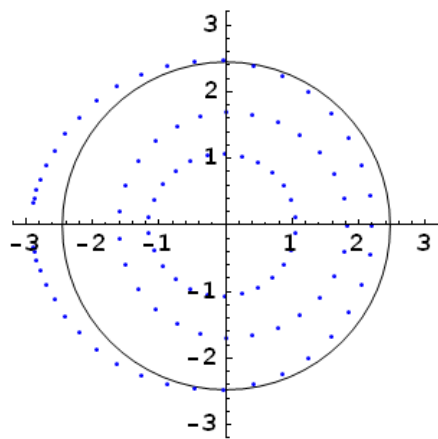


Figure 1.16.11: $\text{Spec}_{4,96}^{\lambda}(\zeta^*)$

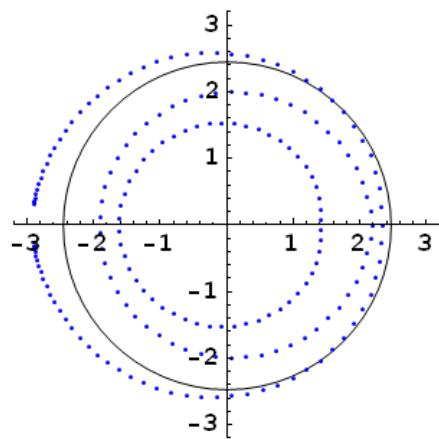


Figure 1.16.12: $\text{Spec}_{4,192}^{\lambda}(\zeta^*)$

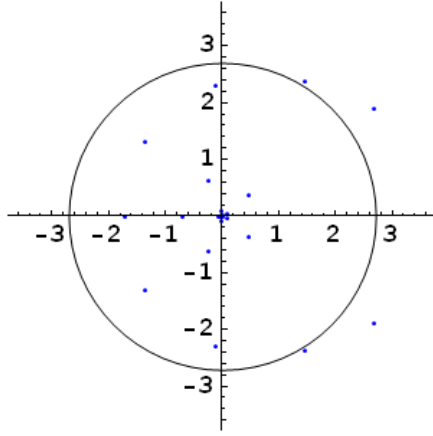


Figure 1.16.13: $\text{Spec}_{5,24}^\lambda(\zeta^*)$

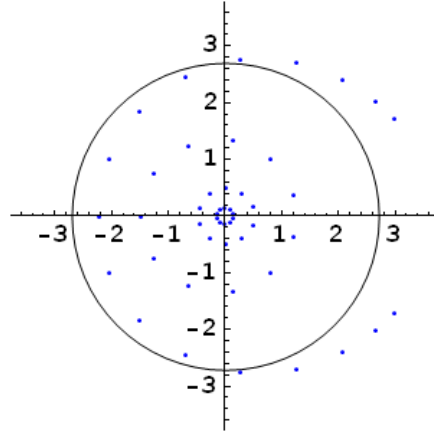


Figure 1.16.14: $\text{Spec}_{5,48}^\lambda(\zeta^*)$

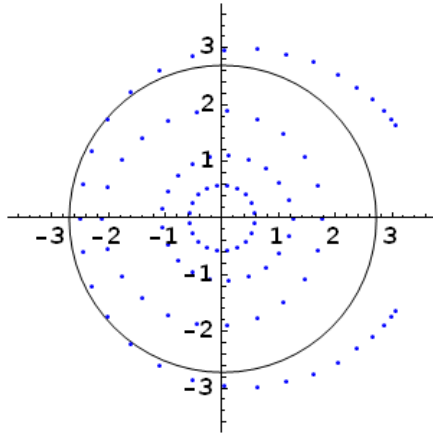


Figure 1.16.15: $\text{Spec}_{5,96}^\lambda(\zeta^*)$

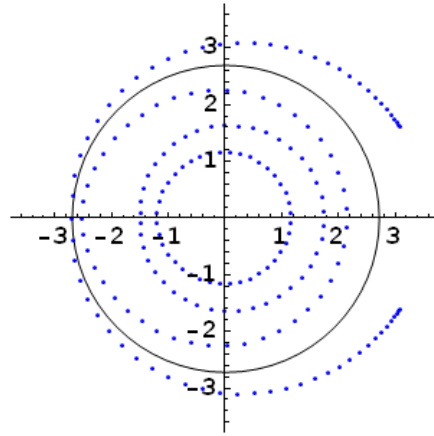


Figure 1.16.16: $\text{Spec}_{5,192}^\lambda(\zeta^*)$

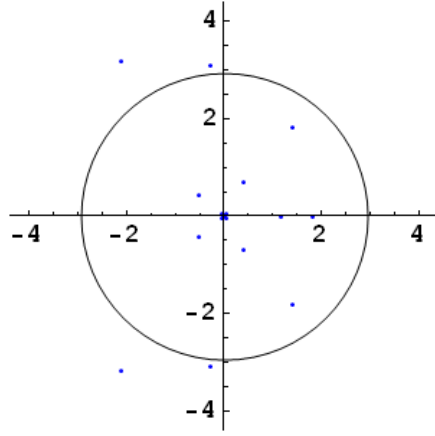


Figure 1.16.17: $\text{Spec}_{6,24}^\lambda(\zeta^*)$

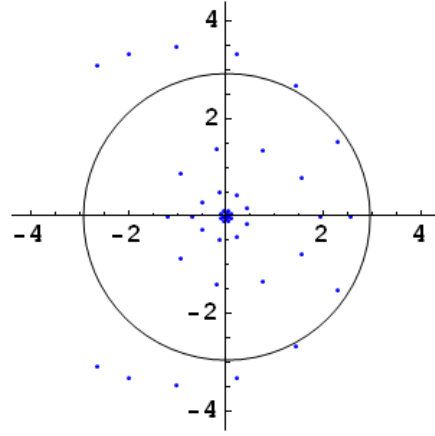


Figure 1.16.18: $\text{Spec}_{6,48}^\lambda(\zeta^*)$

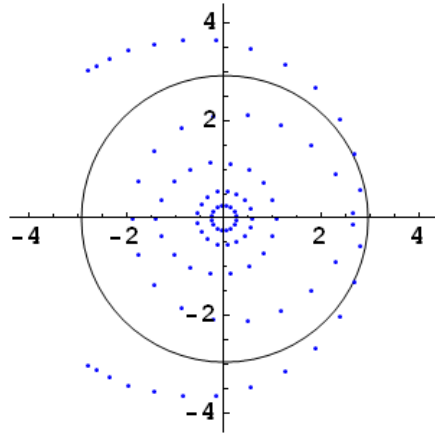


Figure 1.16.19: $\text{Spec}_{6,96}^\lambda(\zeta^*)$

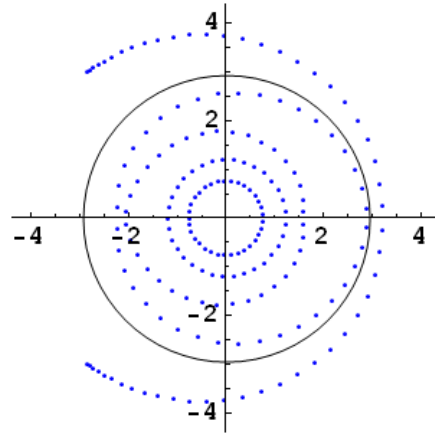


Figure 1.16.20: $\text{Spec}_{6,192}^\lambda(\zeta^*)$

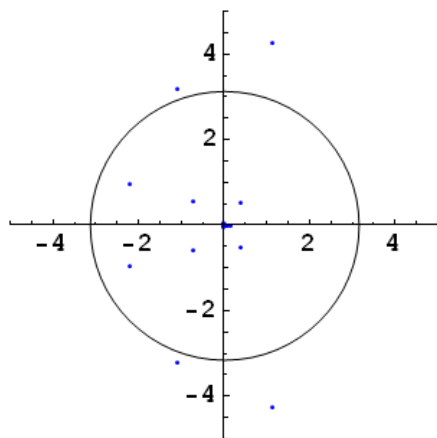


Figure 1.16.21: $\text{Spec}_{7,24}^{\lambda}(\zeta^*)$

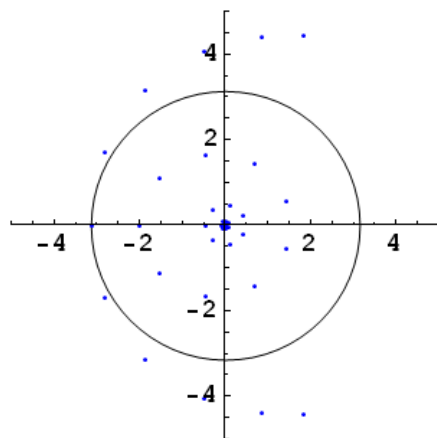


Figure 1.16.22: $\text{Spec}_{7,48}^{\lambda}(\zeta^*)$

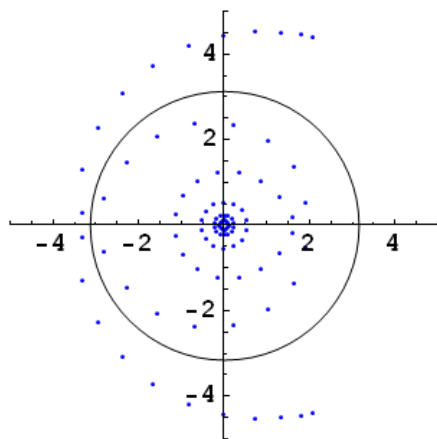


Figure 1.16.23: $\text{Spec}_{7,96}^{\lambda}(\zeta^*)$

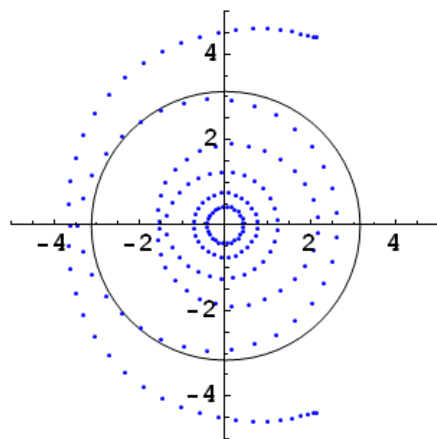


Figure 1.16.24: $\text{Spec}_{7,192}^{\lambda}(\zeta^*)$

Conjecture 1C. For all l , $\inf_m(\min(\text{Arr}_{l,m}(\zeta^*)))$ is a positive number.

Accepting the same agreement about the largest real eigenvalue in $\text{Spec}_{l,m}^\lambda(\zeta^*)$ as was done above in the case of $\text{Spec}_{1,m}^\lambda(\zeta^*)$, we can state the following two conjectures.

Conjecture 1D. For all l and k , the numbers $\text{arr}_{l,m}(\zeta^*) = \|\text{Arr}_{l,m}(\zeta^*)\|$, $\text{orb}_{l,m,k}(\zeta^*) = \|\text{Orb}_{l,m,k}(\zeta^*)\|$, $\text{bow}_{l,m}(\zeta^*) = \|\text{Bow}_{l,m}(\zeta^*)\|$ of eigenvalues belonging to the arrow $\text{Arr}_{l,m}(\zeta^*)$, the orbit $\text{Orb}_{l,m,k}(\zeta^*)$, and the bow $\text{Bow}_{l,m}(\zeta^*)$ respectively don't decrease when m increases.

Conjecture 1E. For all l , if

$$\text{Arr}_{l,m}(\zeta^*) = \{\lambda_{l,m,1}(\zeta^*), \dots, \lambda_{l,m,\text{arr}_{l,m}(\zeta^*)}(\zeta^*)\}, \quad (1.17.1)$$

$$\text{Arr}_{l,m+1}(\zeta^*) = \{\lambda_{l,m+1,1}(\zeta^*), \dots, \lambda_{l,m+1,\text{arr}_{l,m+1}(\zeta^*)}(\zeta^*)\} \quad (1.17.2)$$

and

$$\lambda_{l,m,1}(\zeta^*) < \lambda_{l,m,2}(\zeta^*) < \dots < \lambda_{l,m,\text{arr}_{l,m}(\zeta^*)}(\zeta^*), \quad (1.17.3)$$

$$\lambda_{l,m+1,1}(\zeta^*) < \lambda_{l,m+1,2}(\zeta^*) < \dots < \lambda_{l,m+1,\text{arr}_{l,m+1}(\zeta^*)}(\zeta^*) \quad (1.17.4)$$

then

$$\lambda_{l,m+1,1}(\zeta^*) < \lambda_{l,m,1}(\zeta^*), \dots, \lambda_{l,m+1,\text{arr}_{l,m}(\zeta^*)}(\zeta^*) < \lambda_{l,m,\text{arr}_{l,m}(\zeta^*)}(\zeta^*). \quad (1.17.5)$$

Conjecture 1F. For given l and m , assign the weight $\frac{1}{m}$ to each point $\lambda_{l,m,1}(\zeta^*), \lambda_{l,m,2}(\zeta^*), \dots, \lambda_{l,m,m}(\zeta^*)$ and denote by $\lambda_{l,m}(\zeta^*)$ the corresponding discrete measure. Then

1F'. for $m \rightarrow \infty$ there exists a limiting continuous measure $\lambda_l^{\zeta^*}(w)$ concentrated on a “limiting bow”, a “limiting arrow”, and a “limiting target” consisting of a number of “limiting orbits”;

1F''. $\int \log(w) d\lambda_l^{\zeta^*}(w) = \log(W_l)$.

Clearly, Conjecture 1F implies the Riemann Hypothesis.

An precise definition of the bow $\text{Bow}_{l,m}$ and of individual orbits $\text{Orb}_{l,m,k}$ constituting the target $\text{Targ}_{l,m}$ is a subtle matter because of the rendezvous. The following conjectures implicitly presuppose that *It is possible to split $\text{Spec}_{l,m}^\lambda(\zeta^*)$ into $\text{Arr}_{l,m}(\zeta^*)$, $\text{Orb}_{l,m,k}(\zeta^*)$, and $\text{Bow}_{l,m}(\zeta^*)$ in such a manner that....*

Conjecture 1G. For every l and k the number of eigenvalues in $\text{Arr}_{l,m}(\zeta^*)$, $\text{Orb}_{l,m,k}(\zeta^*)$, and $\text{Bow}_{l,m}(\zeta^*)$ doesn't decrease with the growth of m .

Conjecture 1H. For every $l > 1$, m , and k the eigenvalues from the orbit $\text{Orb}_{l,m,k}(\zeta^*)$ almost lie on a circle and are almost equidistributed on it.

Conjecture 1I. For every $l > 1$ the limiting target $\text{Targ}_l(\zeta^*)$ consists of some number $\text{targ}_l(\zeta^*)$ of limiting orbits $\text{Orb}_{l,k}(\zeta^*)$ and

1I' all limiting orbits are circles;

1I'' on each limiting orbit the limiting measure $\lambda_l^{\zeta^*}(w)$ is constant;

1I''' the limiting orbits $\text{Orb}_{l,k}(\zeta^*)$ can be numbered in such a way that for $k < \text{targ}_l(\zeta^*)$ the limiting orbit $\text{Orb}_{l,k}(\zeta^*)$ lies inside the limiting orbit $\text{Orb}_{l,k+1}(\zeta^*)$ and touches it at one real point $\text{Rend}_{l,k}(\zeta^*)$ called the rendezvous-point; the innermost limiting orbit has rendezvous point $\text{Rend}_{l,0}(\zeta^*)$ with the limiting arrow $\text{Arr}_l(\zeta^*)$, and the outmost limiting orbit has rendezvous point $\text{Rend}_{l,\text{targ}_l(\zeta^*)}(\zeta^*)$ with the limiting bow $\text{Bow}_l(\zeta^*)$; moreover, $\text{Rend}_{l,k+1}(\zeta^*) < \text{Rend}_{l,k}(\zeta^*)$ for even k and $\text{Rend}_{l,k+1}(\zeta^*) > \text{Rend}_{l,k}(\zeta^*)$ for odd k .

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